



## An algorithmic method for showing existence of nontrivial non-classical symmetries of partial differential equations without solving determining equations



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### ABSTRACT

In this paper, based on differential characteristic set theory and the associated algorithm (also called Wu's method), an algorithmic method is presented to decide on the existence of a nontrivial non-classical symmetry of a given partial differential equation without solving the corresponding nonlinear determining system. The theory and algorithm give a partial answer for the open problem posed by P.A. Clarkson and E.L. Mansfield in [21] on non-classical symmetries of partial differential equations. As applications of our algorithm, non-classical symmetries and corresponding invariant solutions are found for several evolution equations.

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## 1. Introduction

The classical symmetry method (CS method), also called the Lie symmetry method, is a powerful general technique for analyzing partial differential equations (PDEs) [7,8,35]. The Lie symmetry method provides the most widely applicable approach to find closed form solutions of PDEs, especially nonlinear ones. Most of the useful systematic methods for solving PDEs such as separation of variables [34], integral transforms [22], self-similar solutions [2] (constructed via reduction by scalings, special first-order symmetry generators), etc., involve direct uses of the Lie symmetry method. Many generalizations of the CS method have been recently developed [29]. One such generalization of the CS method is the non-classical symmetry method (NCS method) for finding further invariant solutions of PDEs [4,6]. There are several different names for methods related to the NCS method, such as the methods involving Q-conditional symmetries, conditional symmetries or reduction operators [24,36,37,52]. In this paper, we consider the original NCS method and use the original term non-classical symmetry [4]. The NCS method is also related to Bäcklund transformations, functionally invariant solutions, and “direct” methods etc. [18].

The calculations involved in the CS method now usually involve routine tasks through the use of symbolic computation software [10,17,28,43]. However, in the non-classical case, it is still a challenging problem since the determining equations are nonlinear unlike the situation for the CS method [21].

Let  $X = (x_1, x_2, \dots, x_p)$  be independent variables and  $U = (u_1, u_2, \dots, u_q)$  be differential real functions of  $X$ . Let  $\partial^\alpha U = \{u_i^\alpha, i = 1, 2, \dots, q\}$ , where  $\alpha = \{\alpha_1, \dots, \alpha_p\} \in Z_+^p$  ( $Z_+$  is the non-negative integer set) and  $u_i^\alpha = u_{i,x_1^{\alpha_1} x_2^{\alpha_2} \dots x_p^{\alpha_p}} = \partial^{|\alpha|} u_i / \partial x_1^{\alpha_1} \dots \partial x_p^{\alpha_p}$  ( $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_p$ ), denote the derivatives of  $U$  with respect to  $X$  and set  $\partial U = \{\partial^\alpha U, \alpha \in Z_+^p\}$ . We write a  $K$ th order PDE system as

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$$F[U] = F(X, \partial^\alpha U) = 0, \quad |\alpha| \leq K, \quad (1)$$

with independent variables  $X$  and dependent variables  $U$ .

A PDE with free parameters usually involves complex symmetry classification problems [7,8,14,29]. For the sake of simplicity, in this paper we only consider a PDE (1) with no free parameters. Nevertheless, there exist also complete classifications of non-classical symmetries. For example, classical and non-classical symmetry classifications for the equations  $u_t = u_{xx} + q(u)$  were given in [1,19,23].

Focusing on practical applications, throughout this paper we only consider PDEs that are non-degenerate in the sense of Olver [35], i.e., a considered system is locally solvable and has maximal rank.

For the convenience of a reader, we give a brief overview of the CS and NCS methods for reducing a given PDE (1).

### 1.1. Classical symmetry

Consider a one-parameter Lie group of transformations in  $(X, U)$ -space given by

$$\begin{aligned} X^* &= X + \varepsilon \xi(X, U) + O(\varepsilon^2), \\ U^* &= U + \varepsilon \eta(X, U) + O(\varepsilon^2), \end{aligned} \quad (2)$$

where  $\varepsilon$  is the group parameter,  $\xi = (\xi_1, \xi_2, \dots, \xi_p)$  and  $\eta = (\eta_1, \eta_2, \dots, \eta_q)$  are the infinitesimals of the group of transformations. The associated Lie algebra element is the infinitesimal generator

$$\mathcal{X} = \xi \cdot \partial_X + \eta \cdot \partial_U, \quad (3)$$

where  $\partial_X = (\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_p})^T$  and  $\partial_U = (\partial_{u_1}, \partial_{u_2}, \dots, \partial_{u_q})^T$ . The prolongation of  $\mathcal{X}$  is given by the following formula [35]

$$\text{Pr } \mathcal{X} = \mathcal{X} + \sum_{k=1}^q \sum_{\alpha} \eta_k^\alpha(X, U^{(K)}) \frac{\partial}{\partial u_k^\alpha},$$

with the second summation being over all (unordered) multi-indices  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$  with  $1 \leq \alpha_k \leq p$ ,  $1 \leq |\alpha| \leq K$ . The coefficient functions  $\eta_k^\alpha$  of  $\text{Pr } \mathcal{X}$  are given by the expressions

$$\eta_k^\alpha = D_\alpha \left( \eta_k - \sum_{i=1}^p \xi_i u_k^{i,\alpha} \right) + \sum_{i=1}^p \xi_i u_k^{\alpha,i},$$

in which  $u_k^i = \partial u_k / \partial x_i$ ,  $u_k^{\alpha,i} = \partial u_k^\alpha / \partial x_i$ ,  $U^{(K)} = \{\partial^\beta u, \text{ for } \beta \in Z_+^p, |\beta| \leq K\}$  and  $D_\alpha$  is the total derivative operator with respect to independent variables  $X$  to order  $\alpha$ .

If PDE (1) is invariant under (2), the group of transformations (2) yields the classical symmetries of the PDE (1). Lie's first fundamental theorem shows the equivalence between determining (2) and (3). The procedure of determining (3) consists of two parts [35]. First, producing the determining equations (DTEs) for the functions  $\xi(X, U)$  and  $\eta(X, U)$ . The DTEs are produced from the invariance criterion of the non-degenerate PDEs (1) under the transformations (2). This is given by  $\text{Pr } \mathcal{X}(F) = 0$  when  $F = 0$ . The set of DTEs is an over-determined linear system of PDEs for the infinitesimals  $\xi, \eta$ ; next, after explicitly solving the DTEs, one obtains (3). In practical computations, each of the two parts often involves a large amount of tedious algebra and auxiliary calculations which can become virtually unmanageable if attempted manually. Hence some symbolic manipulation programs have been developed to help one to obtain (3) mechanically [10,17,28,43].

### 1.2. Non-classical symmetry

In the NCS method, an infinitesimal generator (3) is found by requiring that a given PDE (1) and the equation defined by the invariant surface condition

$$\psi = \xi \cdot U_X - \eta = 0, \quad (4)$$

to obtain corresponding invariant solutions, are simultaneously invariant under transformation (2), where  $U_X = (u_{i,x_j})^T$ . In this case, the procedure for determining (3) is an extension of the classical one and includes the classical procedure as a special case. Here, the invariance criterion is given by the relation  $\text{Pr } \mathcal{X}(F) = 0$  which holds on the manifold defined simultaneously by (1) and the differential consequences of (4). From this criterion, one derives the DTEs for the NCS method. The set of DTEs is an over-determined nonlinear system of PDEs, in contrast to a linear system in the classical case, for the functions  $\xi = \xi(X, U)$  and  $\eta = \eta(X, U)$  that appear in the transformation (2) and the supplementary condition (4). Then, exactly solving the nonlinear DTEs, one finds the generator (3).

In using the invariance criterion, some severe theoretical points, such as the non-degeneracy requirement and reduction with respect to the differential consequences etc. of the PDEs (1) and (4), should be taken into account primarily due to the involved systems being both over-determined and nonlinear. The precise definitions and discussion are presented in [20,52].

Moreover, it is important to mention that there is the concept of “higher” step (order) non-classical symmetries proposed in [38]. In this paper, we only consider showing the existence of non-classical symmetries as proposed in the thesis [4] for a PDE.

Following [32,51], the set of (first-order) differential operators (or generators) of Eq. (1) in general form (3) will be denoted by  $\mathfrak{D}(F)$ . Two differential operators  $\tilde{\mathcal{X}} \in \mathfrak{D}(F)$  and  $\mathcal{X} \in \mathfrak{D}(F)$  are called equivalent if they differ by a multiplier which is non-vanishing function of  $X$  and  $U$ :  $\tilde{\mathcal{X}} = \lambda(X, U)\mathcal{X}$ , where  $\lambda \neq 0$ . The equivalence of operators will be denoted by  $\tilde{\mathcal{X}} \sim \mathcal{X}$ . We use  $\mathfrak{D}_f(F)$  to denote the factoring of  $\mathfrak{D}(F)$  with respect to this equivalence relation. We identify elements of the set of  $\mathfrak{D}_f(F)$  with their representatives in  $\mathfrak{D}(F)$ . In this manner the problem of completely describing all operators (3) for (1) is equivalent to finding  $\mathfrak{D}_f(F)$ . Now, in the equivalence framework, the invariant surface condition (4) is split into  $p$  equivalent cases according to whether the infinitesimal function  $\xi_j$  is zero or nonzero for  $1 \leq j \leq q$ ,

$$\begin{aligned} \xi_1 u_{j,x_1} + \xi_2 u_{j,x_2} + \cdots + \xi_{p-1} u_{j,x_{p-1}} + u_{j,x_p} &= \eta_j, & \text{Case } p, \\ \xi_1 u_{j,x_1} + \xi_2 u_{j,x_2} + \cdots + u_{j,x_{p-1}} &= \eta_j, & \text{Case } p - 1, \\ &\vdots & \\ u_{j,x_1} &= \eta_j, & \text{Case } 1, \end{aligned} \tag{5}$$

i.e., successively for  $1 \leq k \leq p$ ,  $\xi_k$  has been set equal to 1 and  $\xi_{k+1} = \cdots = \xi_n = 0$ . This is directly used in [20]. It is helpful to simplify the computational procedure by reducing the number of unknown functions.

Furthermore, related to the splitting, it is well known that for an evolution equation, non-classical symmetries form “singular” cases [32,51]. In the singular cases, it is not certain that the set of DTEs for non-classical symmetries is over-determined. Hence, its solution set is harder to find for use in symmetry applications. In [32], the notion of singular reduction operators, i.e., of singular operators of non-classical (conditional) symmetries, of PDEs in two independent variables is introduced. Especially for a (1 + 1)-dimensional ( $p = 2$  in (5)) evolution equation of the form  $u_t = H(t, x, u_{(r,x)})$  ( $r$ th order in  $x$ ) as a particular case of the general theory, the singular operator is taken to be of the form  $\mathcal{X} = \partial_x + \eta(t, x, u)\partial_u$  corresponding to non-classical symmetries for the case 1 in (5). In [32,51], the authors prove the one-to-one correspondence between such singular operators and one-parameter families of solutions to the evolution equation. From this, one has the conclusion that if the set of solutions to the DTEs for classical symmetries, in the case of a singular operator, is a proper subset of those for  $\eta(t, x, u)$  for the set of DTEs of the singular operator, then the (1 + 1)-evolution equation admits non-trivial non-classical symmetries. This allows one to determine the existence of nontrivial non-classical symmetries for the case of an evolution equation. This also implies, for the decision problem, one can use the information from the DTEs for classical symmetries to decide on the existence of nontrivial non-classical symmetries. It will be seen that in our proposed method the information from the DTEs for classical symmetries is used to show the existence of nontrivial non-classical symmetries for a given PDE through carrying out the so-called differential characteristic set algorithm (see Remark 2(4) and Remark 3) which fits into more general cases. Hence our proposed method also deals with the singular case.

Additionally, it is well known that the set of classical symmetries is a subset of the set of non-classical symmetries [4, 6,8,18]. A non-classical symmetry which is not a classical symmetry is called a nontrivial non-classical symmetry (NNCS). Thus the focus of the present paper is on the existence of the elements in  $\mathfrak{D}_f(F)$  which satisfy (4) and are not generators of the CSs of (1). However, not all PDEs have an NNCS. Hence, the a priori judgment of whether a given PDE has an NNCS without solving the nonlinear DTEs is of great importance.

### 1.3. An open problem

In [21], Clarkson and Mansfield proposed an open problem: *how to determine a priori which PDEs possess symmetry reductions that are not obtainable using the classical Lie group approach?* In particular, in the NCS case, the problem is equivalent to: *how to know whether a given PDE has an NNCS without solving its nonlinear DTEs?* One also has the critical problem to simplify the procedure of obtaining the NCSs for a PDE. So far, there are few studies on these challenging problems. Related recent works include [11,12,14,15,20,33,39–41,44,47,48]. In [20,33], Clarkson and Mansfield gave an algorithm, based on the use of Gröbner bases, to obtain the DTEs for the NCSs and the corresponding PDE reductions. In the CS case, in [33,39–41,44], Reid, Schwarz etc. investigated the symmetry properties of PDEs without solving the DTEs. In [47,48], Vorob’ev showed that the DTEs for the NCSs admit the CSs of the original equations and hence use the admitted CSs as an aid to find systematically solutions of the DTEs. In [11,12,14,15], we gave an alternative algorithm for determining and classifying symmetries of PDEs based on the differential characteristic set (**dchar-set**) method of a differential polynomial system (**dps**) proposed by Wu [49,50] (also called Wu’s method) and gave many applications of obtaining symmetries and conservation laws for various PDEs [5,11,12,14,15].

In this paper, we present a method to decide whether a given PDE has an NNCS without solving the nonlinear DTEs. Our method is realized by using the dchar-set theory and algorithm [49,50] to obtain the so-called dchar-set decomposition of the dps corresponding to the DTEs for NCSs. The method gives a partial theoretical and practical answer to the open problem mentioned above in the NCS case and to some extent leads to the simplification of the computations for the NCSs.

The rest of this paper is outlined as follows. In Section 2, we briefly overview the dchar-set algorithm of a dps. In Section 3, we give an algorithm to show the existence of an NNCS for a given PDE based on the dchar-set algorithm. Consequently, an algorithm for reducing the DTEs for NCSs is also presented. In Section 4, we give some examples to show

the applicability of the given algorithm. In Section 5, we present some concluding remarks and discussions on related results and algorithms.

**2. Dchar-set algorithm of a dps (Wu’s method)**

For the convenience of a reader, in this section we recall basic results on the dchar-set algorithm of a dps in terms of the independent variables in  $X$ , the depend variables in  $U$  and partial derivative operations  $\partial_{x_i}$  given in Section 1. All concepts on dps are taken from [49] (see also [13,16,25,30,42,50]).

*2.1. Basic results of the dchar-set algorithm*

Let  $\mathcal{K}_X$  be a differential field of characteristic zero of functions of  $X = \{x_1, x_2, \dots, x_p\}$  with derivative operators  $\partial_{x_i}$  ( $i = 1, 2, \dots, p$ ) and let  $\mathcal{K}_X[\partial U]$  be the differential polynomials (**d-pols**) ring in the indeterminate  $\partial U$  over  $\mathcal{K}_X$ . If for a dps  $D \subset \mathcal{K}_X[\partial U]$ , there exists a set of  $q$  elements  $a = \{a_1, a_2, \dots, a_q\}$  over a universal field of  $\mathcal{K}_X$  such that when each  $u_i \in U$  is replaced by  $a_i$  in all the d-pols of  $D$  so that these d-pols reduce to zero, then  $a$  is called a zero of  $D$ . Thus a zero of  $D$  is a solution of the system of equations (simply denoted by  $D = 0$ ) obtained by equating the d-pols in  $D$  to zero. The totality of zeros of  $D$  is called the zero set (i.e., a variety or manifold in [42]) of  $D$  and is denoted by  $\text{Zero}(D)$ . For a dps  $C \subset \mathcal{K}_X[\partial U]$ , let  $\text{Zero}(D/C)$  denote the zero points of  $D$  with  $d_i \neq 0$  for some  $d_i \in C$ , i.e.,  $\text{Zero}(D/C) = \text{Zero}(D) \setminus \text{Zero}(C)$ .

A rank on a d-pol ring in the dchar-set method is an essential point. A natural choice of such a rank is the diff-graded lex order which is induced by the total derivative differential form graded lexicographic rank on derivative terms [30,42,49]. In all examples in this paper, we take the rank as the d-pol rank.

Under a rank, the coefficient of the highest degree term occurring in a d-pol as an ordinary polynomial in the leading derivative is called the initial of the d-pol. The formal partial derivative of a d-pol with respect to the leading derivative is called the separant of the d-pol. Let  $IS$  or  $IS(D)$  denote the product of initials and separants of a dps  $D$  ignoring concrete expressions. For the computations,  $\mathcal{K}_X$  should be considered as a more concrete differential field such as the differential field of meromorphic functions or its various differential subfields.

The concepts of a differential ascending chain and characteristic set of a differential polynomial system play a key role in the dchar-set method. The passive property and cascade (triangular) structure of the characteristic set allow the analysis of the zero set of the dps (equivalently, the solution set of the corresponding PDEs) to be more convenient.

We use  $I$  to denote the integrability (coherent) polynomial (condition) obtained from any two differential polynomials and use  $\text{Prem}(dp/dq)$  to denote the pseudo-remainder of the differential polynomial  $dp$  with respect to the differential polynomial  $dq$ . In Wu’s method, the pseudo-remainder operation originates from Ritt’s elimination procedure [42].

The outline of the differential characteristic set algorithm for a dps  $DP$  is as follows.

**Algorithm A** (*Dchar-set algorithm*).

**Input:** A dps  $DP$ .

**Output:** A dchar-set  $DCS$  of the dps  $DP$ .

**Begin:** Let  $i = 0, DP_0 = DP$ .

**Step 1.** Choose a base set  $BS_i$  (the lowest rank differential ascending set) of  $DP_i$ .

**Step 2.** Compute all integrality polynomials  $I$  of  $BS_i$  in its completion set.

**Step 3.** Compute  $IT_i = \text{Prem}(I/BS_i) \setminus \{0\}$  for all  $I$  obtained in Step 2.

**Step 4.** Compute  $R_i = \text{Prem}((DP_i \setminus BS_0)/DBS_i) \setminus \{0\}$  and let  $RI_i = IT_i \cup R_i$ .

**Step 5.** If  $RI_i = \emptyset$ , let  $DCS = BS_i$  and stop the computation; else let  $i = i + 1$  and  $DP_i = DP_{i-1} \cup BS_{i-1} \cup RI_{i-1}$  and return to Step 1.

**End.**

The above procedure can be illustrated by the following scheme for intuitive understanding.

$$\begin{array}{ccccccc}
 DP = DP_0 & \rightarrow & DP_1 & \rightarrow & \cdots & \rightarrow & DP_{s-1} & \rightarrow & DP_s \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 BS_0 & \succcurlyeq & BS_1 & \succcurlyeq & \cdots & \succcurlyeq & BS_{s-1} & \succcurlyeq & BS_s = DCS \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 RI_0 & & RI_1 & & \cdots & & RI_{s-1} & & RI_s = \emptyset.
 \end{array} \tag{W}$$

The down arrow means that the computation is continuous in this step and the right arrow shows the computation turning into the next loop step.

Wu proved the following basic result [49].

**Theorem 1** (*Differential characteristic set*). For a differential polynomial system  $DP$ , **Algorithm A** or the scheme (W) will end in a finite number of steps at some stage  $s$  with corresponding remainder set  $RI_s = \emptyset$ . The corresponding base set  $DCS = BS_s$ , called the differential characteristic set of the dps  $DP$ , verifies the properties  $(a_1)$ ,  $(a_2)$  and  $(a_3)$  below:

- (a<sub>1</sub>)  $\text{Prem}(DP/DCS) = 0$ .
- (a<sub>2</sub>)  $\text{Zero}(DP) \subset \text{Zero}(DCS)$ .
- (a<sub>3</sub>)  $\text{Prem}(I/DCS) = 0$  for all integrability polynomials  $I$  of  $DCS$ .

Algorithm A or the scheme (W) and Theorem 1 are given by Wentsun Wu, called Wu’s elimination algorithm, which are essential parts of Wu’s method. It has now become a fundamental basis of differential characteristic set theories and algorithms. See the original paper [49] for the details of the method (see also [13,16,26]). The algorithm is implemented in [11] with Mathematica by the first author of the present paper and in [26] with Maple by Gao and Wang as part of the computer algebra system MMP. In this paper, we use the package in [11] in the calculations for all examples.

Now we delineate further the basic results of the dchar-set theory which are obtained based on Algorithm A and used in this paper.

**Theorem 2** (Wu’s well-ordering principle). (See [49].) For a given finite set differential polynomial system  $DP$ , Algorithm A yields a dchar-set  $DCS$  of the differential polynomial system in a finite number of steps for the well-ordering principle

$$\begin{aligned} \text{Zero}(DCS/IS) &\subset \text{Zero}(DP) \subset \text{Zero}(DCS), \\ \text{Zero}(DP) &= \text{Zero}(DCS/IS) \cup \text{Zero}(DP, IS), \end{aligned} \tag{6}$$

holding true, where  $IS$  is the product of initials and separants of the  $DCS$ .

**Theorem 3** (Ritt–Wu’s zero-decomposition theorem). (See [49].) For a given finite set differential polynomial systems  $DP$ , there is an algorithm which allows zero decompositions of the form

$$\text{Zero}(DP) = \bigcup_k^N \text{Zero}(C_k/IS_k), \tag{7}$$

in a finite number of steps, in which  $C_k$  is an (irreducible) passive differential ascending chain (in fact, it is an (irreducible) dchar-set of an extended differential polynomial system obtained by adding some differential polynomials in  $DP$ ) and  $IS_k$  is the production of initials and separants of  $C_k$  and  $N$  is a positive integer.

The algorithm in the theorem is a procedure of repeatedly applying Algorithm A to the part  $\text{Zero}(DP, IS)$  of the right hand side of (6).

Based on Wu’s zero decomposition (7), one has the following theorem.

**Theorem 4.** If the zero set  $\text{Zero}(DP')$  is irreducible and contained in  $\text{Zero}(DP)$  which has a decomposition as in the above theorem, then  $\text{Zero}(DP')$  is contained in some  $\text{Zero}(C_i)$ .

**Proof.** Since  $\text{Zero}(C_k/IS_k) \subseteq \text{Zero}(C_k)$ , one has  $\text{Zero}(DP') \subseteq \bigcup_k^N \text{Zero}(C_k)$ . Hence the theorem is proved by Proposition 2.4 in [50] or from the conclusion in [42] (see pp. 23–24, Section 3, Chapter II). □

For a quasi-variety, the analogue of Wu’s Theorem 3 was given in [25] as follows:

**Theorem 5.** (See [25].) For two finite differential polynomial systems  $DP$  and  $DP'$ , there is an algorithm that shows either  $\text{Zero}(DP/DP') = \emptyset$  or finds a passive differential ascending  $C_i$  and some positive integer  $N$  such that

$$\text{Zero}(DP/DP') = \bigcup_i^N \text{Zero}(C_i/IS_i * DP'), \tag{8}$$

in a finite number of steps where  $IS_i$  is the production of initials and separants of  $C_i$ .

The algorithm in the theorem is designed based on Algorithm A under the condition  $DP' \neq 0$ . The significance of this theorem will be seen in our method developed in the following section.

### 2.2. An example for the dchar-set algorithm

In the following, we give an illustrative example to show the well-ordering (cascade) structure of a dchar-set and its effective application in solving an over-determined system. In order to see this clearly, we take a variable coefficients linear dps as an example.

**Example.** For the classical symmetries of the extended system of nonlinear wave equations [5],

$$u_t - v_x = 0, \quad v_t - e^u u_x - e^u = 0, \quad p_t - e^u (q_x - q) = 0, \quad p_x - q_t = 0,$$

one has the set of DTEs  $DP = 0$ , where the left hand side of the DTEs

$$DP = \left\{ \begin{array}{l} \xi_q - \tau_p, e^u \xi_q - \eta_p + e^u \tau_p, \xi_p - e^u \tau_q, \zeta_p - \eta_q + e^u \tau_q, \\ \xi_u - e^u \tau_v, pe^u \tau_v + \phi_v - \phi_u, \xi_v - \tau_u, pe^u \tau_u + \phi_u - e^u \phi_v, \\ \zeta_u - \eta_v + \xi_x + pe^u \tau_q - \tau_t, e^u \tau_v - \xi_x + \tau_t + \phi_q - \phi_p, \\ \zeta + \zeta_u - \eta_v - \xi_x - pe^u \tau_q - 2e^u \tau_v + \tau_t, pe^u \tau_x + \phi_x + pe^u \phi_q - e^u \phi_v - \phi_t, \\ e^u \zeta + e^u \xi_x + pe^u \eta_q - e^u \eta_v - \eta_t - pe^{2u} \tau_q + e^{2u} \tau_v + e^u \tau_t, \\ e^u \zeta - e^u \xi_x - 2pe^{2u} \tau_q + e^{2u} \tau_v + e^u \tau_t - e^u \phi_q + e^u \phi_p, \\ p^2 e^{2u} \tau_q - pe^u \zeta - pe^{2u} \tau_v - pe^u \tau_t + e^u p \phi_q - e^u \phi_v - \phi_t - e^u \phi + e^u \phi_x, \\ e^u \xi_v - pe^u \xi_q + \xi_t - pe^u \tau_p - e^u \tau_x - \phi_p + e^u \phi_q, \\ e^u \xi_v - pe^u \xi_q + \xi_t + pe^u \tau_p - e^u \tau_x + \phi_p - e^u \phi_q, \\ pe^u \zeta_q - e^u \zeta_v - \zeta_t + \eta_x - e^u \tau_x, e^u \zeta_v - \eta_u - pe^u \xi_q + e^u \xi_v + \xi_t + e^u \tau_u - e^u \tau_x, \\ e^u \zeta_v - \eta_u + pe^u \xi_q - e^u \xi_v - \xi_t + e^u \tau_u + e^u \tau_x \end{array} \right\}$$

is the dps in  $\mathcal{K}_X[\partial U]$  with  $X = (x, t, u, v, p, q)$  and  $U = (\xi, \zeta, \tau, \eta, \phi, \varphi)$  corresponding to the classical symmetries.

Under the basic ranking  $x < t < u < \xi < \zeta < \tau < \eta < \phi < \varphi$ , applying Algorithm A on  $DP$ , one obtains the dchar-set  $DCS$  of the dps  $DP$  as follows:

$$DCS = \left\{ \begin{array}{ll} \xi_p, \xi_q, \xi_v, \xi_u, \xi_t, \xi_x; & \phi_{qq}, \phi_{uq}, \phi_{tq}, \phi_{tt}, \phi_{xq}, \phi_p, \phi_u - \phi_{uu}, \\ \zeta_p, \zeta_q, \zeta_v, \zeta_u, \zeta_t, \zeta_x; & e^u \phi_v - \phi_t, \phi_t - \phi_{tu}, \phi_u - \phi_{xu}, \phi_t - \phi_{xt}, \phi_x - \phi_{xx}; \\ \tau_p, \tau_q, \tau_v, \tau_u, \tau_x, 2\tau_t + \zeta; & \varphi_q, \zeta - 2\phi_q + 2\varphi_p, \phi_u - e^u \phi_v, \phi_t - e^u \phi_u, \\ \eta_p, \eta_q, \eta_u, \eta_t, \eta_x, 2\eta_v - \zeta; & \phi_u - \phi_x + \varphi_t, e^u p \zeta - 2e^u p \phi_q + 4\phi_t + 2e^u \varphi - 2e^u \varphi_x \end{array} \right\}.$$

Since  $IS = -2e^u \in \mathcal{K}_X (\neq 0)$ , one has

$$\text{Zero}(DP) = \text{Zero}(DCS),$$

from the second equality in Theorem 2. This shows the equivalence in solving  $DP = 0$  and  $DCS = 0$ . The well-ordering (cascade form) structure of the dchar-set  $DCS = 0$  is seen from its six parts. The first two parts consist of the first two lines of 12 d-pols just involving  $\xi$  and  $\zeta$  separately. The third part is the next six d-pols only involving  $\zeta$  and  $\tau$ . The fourth part is the following six d-pols involving  $\zeta$  and  $\eta$ . The fifth part is the following 12 d-pols involving  $\phi$  only. The sixth part is the last six d-pols involving  $\zeta, \phi$  and  $\varphi$ . Obviously, the equivalence and well-ordering (triangular) structure of the dchar-set  $DCS$  allow the determination of  $\text{Zero}(DP)$  to be easier through solving  $\text{Zero}(DCS)$ . The infinitesimals  $\xi$  and  $\zeta$  are obtained from the first two parts of  $DCS$ ; the infinitesimals  $\tau, \eta, \phi, \varphi$  are obtained sequentially from the following parts of the  $DCS$  through using the previously obtained zero sets step-by-step. Finally, one has

$$\text{Zero}(DP) = \left\{ \begin{array}{l} \xi = c_1, \zeta = c_2, \\ \tau = -\frac{1}{2}c_2 t + c_4, \eta = \frac{1}{2}c_2 v + c_3, \\ \phi = c_5 q + c_6(e^u t + v)e^x + c_7, \\ \varphi = \frac{1}{2}(2c_5 - c_2)p + c_6(tv + u + 2x)e^x \end{array} \right\}.$$

### 3. An algorithm on the existence of a nontrivial non-classical symmetry

Let  $\xi' = (\xi'_1, \xi'_2, \dots, \xi'_p)$ ,  $\eta' = (\eta'_1, \eta'_2, \dots, \eta'_q)$  and  $\xi = (\xi_1, \xi_2, \dots, \xi_p)$ ,  $\eta = (\eta_1, \eta_2, \dots, \eta_q)$  be the infinitesimals of classical and non-classical symmetries, respectively. Let  $D' = 0$  and  $D = 0$ , respectively, denote the sets of DTEs for the CSs and NCSs of PDE (1). We call the left hand sides  $D' \subset \mathcal{K}_{XUU}[\xi'_i, \eta'_j]$  and  $D \subset \mathcal{K}_{XUU}[\xi_i, \eta_j]$  of the DTEs as the corresponding dps of the CSs and NCSs respectively. Consequently, the sets  $\text{Zero}(D')$  and  $\text{Zero}(D)$  represent the respective solution sets of the DTEs, i.e., the sets of all infinitesimals for the CSs and NCSs of the PDE.

#### 3.1. Theoretical results

For a PDE containing no free parameters, the coefficients of the linear d-pols in  $D'$  are in the basic field  $\mathcal{K}_{XUU}$ . Hence its  $IS$  (the product of initials and separants of  $D'$ ) also belongs to  $\mathcal{K}_{XUU}$ . This implies  $IS \neq 0$ . Thus, one has the following theorem using (6) in Theorem 2 (see also [14]).

**Theorem 6.** For the differential polynomial system  $D'$  corresponding to a classical symmetry of a PDE (1), one has

$$\text{Zero}(D') = \text{Zero}(C'), \tag{9}$$

where  $C'$  is the dchar-set of  $D'$  obtained by Algorithm A.



This theorem yields a simpler procedure for determining the classical symmetries of a PDE. Since  $C'$  admits a cascade structure, it is solved more easily than the original system  $D' = 0$  (see the example given in the previous section).

One bears in mind the fact that the set of classical symmetries is a subset of the set of non-classical symmetries, i.e., for the classical symmetries, the relations

$$\xi' = a\xi, \quad \eta' = a\eta, \tag{10}$$

hold for a common nonzero factor  $a$  and, in general,

$$\text{Zero}(D') \subseteq \text{Zero}(D). \tag{11}$$

The following basic criteria are immediately observed from (9) and (11).

**Theorem 7.** For a PDE (1), let  $C'$  denote the differential characteristic set of the differential polynomial system  $D'$  corresponding to its classical symmetries.

1. The necessary and sufficient condition for the PDE to have a nontrivial non-classical symmetry is given by  $\text{Zero}(C') \subset \text{Zero}(D)$ ; equivalently,

$$\text{Zero}(D/C') \neq \emptyset. \tag{12}$$

2. If the PDE admits a nontrivial non-classical symmetry, then  $\text{Zero}(D/C')$  yields all of them.

**Proof.** The first conclusion is obvious from (11). From  $\text{Zero}(D) = \text{Zero}(D/D') \cup \text{Zero}(D')$ , the second conclusion also holds.  $\square$

From Theorem 3, for  $DP = D$ , one has

$$\text{Zero}(D) = \bigcup_{i=1}^N \text{Zero}(C_i/IS_i), \tag{13}$$

where  $C_i$  is an irreducible dchar-set and  $IS_i$  is the production of initials and separants of  $C_i$ .

Thus one has the following theorem.

**Theorem 8.** For a PDE (1), the differential polynomial system  $D$  corresponding to its non-classical symmetries admits the decomposition (13) and there exists a positive integer  $1 \leq i_0 \leq N$  such that

$$\text{Zero}(D') \subseteq \text{Zero}(C_{i_0}).$$

**Proof.** Due to the linearity of  $D'$  and (11), the conclusion of the theorem follows immediately from Theorem 4.  $\square$

**Remark 1.** Theorem 7 shows that one can make a judgment on the existence of nontrivial non-classical symmetries by combining the DTEs for classical and non-classical symmetries. Theorem 8 shows that the set of classical symmetries is focused on the right hand component of the decomposition (13). Hence it is possible that one can extract the solutions representing the classical symmetries from the solution set of the DTEs  $D = 0$  for the non-classical symmetries before solving the nonlinear DTEs. This leads to a reduction of the DTEs  $D = 0$  and makes it easier to completely solve the DTEs. The extraction of the classical symmetries can be realized by eliminating the equations whose solutions represent the classical symmetries in the procedure for obtaining the decomposition (13) or the decomposition

$$\text{Zero}(D/C') = \bigcup_i \text{Zero}(C_i/IS_i * C'), \tag{14}$$

through the use of the algorithms given in Theorem 3 or 5.

### 3.2. Algorithm

We now present our algorithm. First, for the quasi-variety  $\text{Zero}(D/C')$ , we use Theorem 5 and its algorithm to obtain the decomposition (13) or (14). Then we check the conclusions of Theorem 7 to enable us to determine the existence of a nontrivial non-classical symmetry for a given PDE without solving its DTEs. Our algorithm is given according to the cases listed in (5). Due to the similarity of the cases, we take the case  $p$  in (5) as the example to exhibit our deduction and algorithm.

Since  $\xi_p \equiv 1$  (hence  $\xi'_p \neq 0$  from (10)), the infinitesimal generator for a non-classical symmetry is written as

$$\mathcal{X} = \partial_{x_p} + \xi \cdot \partial_{X'} + \eta \cdot \partial_U, \tag{15}$$

where  $X' = (x_1, x_2, \dots, x_{p-1})$ ,  $\partial_{X'} = (\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_{p-1}})^T$  and  $\xi = (\xi_1, \xi_2, \dots, \xi_{p-1})$ ,  $\eta = (\eta_1, \eta_2, \dots, \eta_q)$  are the new infinitesimals. As mentioned previously,  $C'$  denotes the dchar-set of  $D'$  corresponding to the classical symmetries. In this case, since there are different unknowns in  $D'$  ( $p + q$  unknowns  $(\xi', \eta')$ ) and  $D$  ( $p + q - 1$  unknowns  $(\xi, \eta)$ ), the application of the dchar-set algorithm for the decomposition (14) of  $\text{Zero}(D/C')$  becomes indirect.

The definition of the set  $\text{Zero}(D/C')$  is reformulated as

$$\text{Zero}(D/C') = \{(\xi, \eta) \in \text{Zero}(D) \mid (\xi \cup \xi'_p, \eta) \notin \text{Zero}(C') \text{ for any } \xi'_p\}$$

where  $\xi \cup \xi'_p = (\xi_1, \dots, \xi_{p-1}, \xi'_p)$ . So in order to realize the dchar-set algorithm on the decomposition (14), we need to transfer  $C' \subseteq \mathcal{K}_{X \cup U}[\xi', \eta']$  into a dps in  $\mathcal{K}_{X \cup U}[\xi, \eta]$  having the same unknowns as the dps  $D$ .

The form of (15) shows that for a classical symmetry the relations (10) become

$$\xi'_i = \xi'_p \xi_i, \quad i = 1, 2, \dots, p - 1, \quad \eta'_j = \xi'_p \eta_j, \quad j = 1, 2, \dots, q. \tag{16}$$

Then substituting the relations (16) into the dchar-set  $C'$  of the dps  $D'$ , one transfers the dps  $C' \subseteq \mathcal{K}_{X \cup U}[\xi', \eta']$  into the dps  $C' \subseteq \mathcal{K}_{X \cup U}[\xi \cup \xi'_p, \eta]$ . To eliminate  $\xi'_p$  as soon as possible from  $C'$ , we compute the dchar-set of the transferred  $C'$  under the d-pol rank obtained from the basic ranking by adding to it the unknown  $\xi'_p$  that has the highest rank. Thus we have the decomposition

$$\begin{aligned} \text{Zero}(C') &= \text{Zero}(C''/IS) \cup \text{Zero}(C', IS), \\ \text{Zero}(C''/IS) &\subseteq \text{Zero}(C') \subseteq \text{Zero}(C''), \end{aligned} \tag{17}$$

from Theorem 2, where  $C''$  is the dchar-set of  $C'$  as a dps in  $\mathcal{K}_{X \cup U}[\xi \cup \xi'_p, \eta]$  and  $IS$  is a production of initials and separants of  $C''$ .

Now, one can prove the following theorem.

**Theorem 9.** *Let  $C'$  and  $C''$  be the same as mentioned above. Then*

$$\text{Zero}(D/C'') \subseteq \text{Zero}(D/C') \subseteq \text{Zero}(D/C'') \cup \text{Zero}(D, IS). \tag{18}$$

**Proof.** The proof follows immediately from (17).  $\square$

From Theorem 9, one observes that each nontrivial non-classical symmetry of a PDE ‘almost’ belongs to  $\text{Zero}(D/C'')$  through the difference set  $\text{Zero}(D, IS)$ . Thus it is possible to determine the existence of an NNCS for a given PDE through the decomposition of  $\text{Zero}(D/C'')$  and  $\text{Zero}(D, IS)$  in terms of Zero sets of dchar-sets.

Let  $C$  be the set of all d-pols in  $C''$  whose lead terms are free of  $\xi'_p$ . Let  $\tilde{C} = C'' \setminus C$ .

Moreover, one has two additional criteria for the existence of an NNCS for a given PDE, namely through excluding either  $\text{Zero}(C'')$  or  $\text{Zero}(C)$  from  $\text{Zero}(D)$ .

**Corollary 1.** *If  $IS \neq 0$ , then a given PDE has a nontrivial non-classical symmetry if and only if  $\text{Zero}(D/C'') \neq \emptyset$ .*

**Proof.** Since under condition  $IS \neq 0$ , one has  $\text{Zero}(D, IS) = \emptyset$ . Hence, the proof of the corollary is obtained immediately from (18) and Theorem 7.  $\square$

**Corollary 2.** *Suppose for a given PDE, one has  $\text{Zero}(D/C) \neq \emptyset$ . Then the PDE has a nontrivial non-classical symmetry.*

**Proof.** The conclusion follows from (18) and from  $\text{Zero}(D/C) \subseteq \text{Zero}(D/C'')$  since  $\text{Zero}(C'') \subseteq \text{Zero}(C)$ .  $\square$

**Remark 2.** (1) Since the dps  $D$  is a dps free of  $\xi'_p$ , the mechanism of the dchar-set algorithm demonstrates that the d-pols in  $\tilde{C}$  are not involved in the procedure of obtaining the decomposition (14) of  $\text{Zero}(D/C'')$  in terms of dchar-sets [25,49]. Therefore, in the computation of decomposition (14) for  $\text{Zero}(D/C'')$  in (18), one can replace  $C''$  by  $C$  so that the computation can be completed in the ring  $\mathcal{K}_{X \cup U}[\xi, \eta]$  instead of the mixed ring  $\mathcal{K}_{X \cup U}[\xi \cup \xi'_p, \eta]$ .

(2) Since  $IS(C'')$  consists of the leading coefficients of the d-pols in  $C'' \subseteq \mathcal{K}_{X \cup U}[\xi \cup \xi'_p, \eta]$ , the set  $\text{Zero}(D, IS)$  in (18) is determined easily through the dchar-set algorithm.

(3) In most practical cases, one has  $IS \neq 0$ . Hence, the conclusion of Corollary 1 is often of use in applications.

(4) In the ‘singular’ cases of non-classical symmetries [32], since the dps corresponding to the set of DTEs for classical symmetries is concise and simple (since some  $\xi_j = 0$ ), Corollaries 1–2 are checked easily. Here, we use the differential



polynomial systems  $C$  and  $C''$  obtained from the set of DTEs for the classical symmetries to reduce the dps  $D$  corresponding to the DTEs for non-classical symmetries. Hence, our proposed algorithm deals also with singular non-classical symmetries. See the examples in Section 4 with the  $(\tau, \xi) = (0, 1)$  case and Remark 3.

Summarizing the above procedure and combining Theorem 9, Corollaries 1–2 and Remark 2, one obtains the following algorithm to decide on the existence of a nontrivial non-classical symmetry for a given PDE without solving the corresponding DTEs for each case in (5). Moreover, one has a reduction of the DTEs for the non-classical symmetries of the PDE through decomposition (14).

**Algorithm B.** To decide on the existence of a nontrivial non-classical symmetry for a PDE (1) and yield a reduction of the DTEs for the nontrivial non-classical symmetry.

**Input:** A PDE with a d-pol rank  $\prec$ .

**Output:** Either ‘Yes’ when admitting an NNCS and dchar-sets  $C_i$  (which are the reduction systems of  $D$ ) of the dps  $D$  corresponding to the DTEs  $D = 0$  for an NCS of the PDE with corresponding degenerate conditions  $IS_i$  of the related dchar-sets or ‘No’ when not admitting any NNCS.

**Begin:**

- Step 1.** Produce DTEs  $D' = 0$  and  $D = 0$  for classical and non-classical symmetries, respectively (by Lie’s algorithm).
- Step 2.** Calculate the dchar-set  $C'$  of  $D'$  (by the dchar-set algorithm).
- Step 3.** Repeat for  $i = p$  to 1 do
  - Step 3.1.** Let  $C' :=$  the transformation of  $C'$  by (16).
  - Step 3.2.** Calculate the dchar-set  $C''$  of  $C'$  and obtain  $IS_i = IS(C'')$  under the ranking obtained from the basic ranking  $\prec$  by adding to it the unknown  $\xi'_i$  with the highest ranking (by the dchar-set algorithm).
  - Step 3.3.** Take  $C :=$  the set of all d-pols of  $C''$  with lead terms in  $\mathcal{K}_{X \cup U}[\xi, \eta]$ .
  - Step 3.4.** If  $IS_i \neq 0$  (equivalently  $IS_i \in \mathcal{K}_{X \cup U}$ ), then set  $DI_i = \emptyset$ ; otherwise compute  $DI_i = \text{Zero}(\{D, IS_i\}) = \bigcup_j \text{Zero}(\tilde{C}_{ij}/\tilde{IS}_{ij})$  (by the dchar-set algorithm in Theorem 5).
  - Step 3.5.** Calculate  $\text{Zero}(D/C) = \bigcup_j \text{Zero}(C_j/IS_j)$  (by the dchar-set algorithm in Theorem 5).
  - Step 3.6.** If  $\text{Zero}(D/C) \neq \emptyset$ , then return ‘Yes’ and set  $DZ_i = \bigcup_j \text{Zero}(C_j/IS_j)$ .
  - Step 3.7.** If  $\text{Zero}(D/C) = \emptyset$  and  $DI_i = \emptyset$ , then return ‘No’.
  - Step 3.8.** If  $\text{Zero}(D/C) = \emptyset$  and  $DI_i \neq \emptyset$ , then return ‘No’ if  $DI_i \subseteq \text{Zero}(C')$ ; otherwise return ‘Yes’ (by the dchar-set algorithm in Theorem 3).
  - Step 3.9.**  $i := i - 1$  go to Step 3.1.

**End.**

Finally, to determine an explicit nontrivial non-classical symmetry of the PDE, one exactly solves  $DZ_i$  and  $DI_i$  for  $i = 1, 2, \dots, p$ .

#### 4. Examples

In this section, we apply our algorithm to several examples to illustrate its effectiveness.

**Example 1 (Burgers equation).** As a first example, we consider the problem of finding non-classical symmetries for the Burgers equation  $\Delta = u_t + uu_x + u_{xx} = 0$ . This problem was considered in [15,20,29]. Here, through our algorithm, we obtain the solution from a different point of view. Let

$$\mathcal{X}' = \tau'(x, t, u)\partial_t + \xi'(x, t, u)\partial_x + \eta'(x, t, u)\partial_u, \tag{19}$$

be the infinitesimal generator of a CS.

For showing the NNCS  $\mathcal{X}' = \tau(t, x, u)\partial_t + \xi(x, t, u)\partial_x + \eta(x, t, u)\partial_u$  for the equation, we consider the two equivalent cases (see [32]) in  $\tau \neq 0$  and  $\tau = 0, \xi \neq 0$  independently in the following.

**Case  $\tau \neq 0$ .**

In this case, we let  $\tau \equiv 1$  and have the invariant surface condition

$$u_t - \eta(x, t, u) + \xi(x, t, u)u_x = 0. \tag{20}$$

Executing our Algorithm B, we have the following procedure.

**Step 1.** By the standard procedure given in [4,6,8], one obtains the DTEs  $D' = 0$  for  $\mathcal{X}'$  and the DTEs  $D = 0$  for  $\mathcal{X}$ . Here

$$D' = \{\eta'_{xx}, \xi'_u, \xi'_{xx}, \tau'_u, \tau'_x, 2\xi'_x - \tau'_t, \eta'_t + u\eta'_x, \eta'_u + \xi'_x, -\xi'_t + u\xi'_x + \eta'\}$$

and

$$D = \{\xi_{uu}, \eta_t + u\eta_x + \eta_{xx} + 2\eta\xi_x, \eta_{uu} + 2u\xi_u - 2\xi\xi_u - 2\xi_{xu}, 2\eta_{xu} + 2\eta\xi_u - \xi_t + u\xi_x - 2\xi\xi_x - \xi_{xx} + \eta\}.$$

Now set  $X = (x, t, u)$ ,  $U' = (\xi', \eta', \tau')$  and  $U = (\xi, \eta, \tau)$  and use the basic ranking  $x < t < \xi' < \eta' < \tau'$  on  $\mathcal{K}_X[\partial U']$ .

**Step 2.** Applying Algorithm A to  $D'$ , one obtains the dchar-set  $C'$  of  $D'$  as follows

$$C' = \{ \eta'_{xx}, \eta'_t + u\eta'_{x'} + \xi'_u, \xi'_{xx}, \eta'_u + \xi'_{x'}, \eta' - \xi'_t + u\xi'_{x'}, \tau'_u, \tau'_{x'}, 2\xi'_u - \tau'_t \}.$$

**Step 3.** Let  $\xi'_p = \tau'$ .

**Step 3.1.** Using the transformations  $\xi' = \tau'\xi$  and  $\eta' = \tau'\eta$  into  $C'$ , one gets

$$C' = \{ \eta_{xx}\tau' + \eta\tau'_{xx} + 2\eta_x\tau'_x, \eta_t\tau' + \eta\tau'_t + u(\eta_x\tau' + \eta\tau'_x), \xi_u\tau' + \xi\tau'_u, \eta_u\tau' + \eta\tau'_u + \xi_x\tau' + \xi\tau'_x, \\ u(\xi_x\tau' + \xi\tau'_x) - \xi_t\tau' - \xi\tau'_t + \eta\tau', \xi_{xx}\tau' + \xi\tau'_{xx} + 2\xi_x\tau'_x, \tau'_u, 2(\xi_x\tau' + \xi\tau'_x) - \tau'_t, \tau'_x \}.$$

**Step 3.2.** Calculating the dchar-set  $C''$  of  $C'$  through the dchar-set algorithm under the ranking  $x < t < u < \xi < \eta < \tau'$  on  $\mathcal{K}_X[\partial U]$ , one has

$$C'' = \{ \eta_{xx}, \xi_{xx}, \xi_u, \eta_u + \xi_x, \eta_t + u\eta_x + 2\xi_x\eta, u\xi_x - \xi_t - 2\xi_x\xi + \eta, \tau'_u, 2\xi_x - \tau'_t, \tau'_x \},$$

with  $IS(C'') = \pm 1$ .

**Step 3.3.** Take

$$C = \{ \eta_{xx}, \xi_{xx}, \xi_u, \eta_u + \xi_x, \eta_t + u\eta_x + 2\xi_x\eta, u\xi_x - \xi_t - 2\xi_x\xi + \eta \}.$$

**Step 3.4.** Since  $IS \neq 0$ , set  $DI = \emptyset$ .

**Step 3.5.** Calculating  $\text{Zero}(D/C)$  by the algorithm given in Theorem 4, one obtains

$$\text{Zero}(D/C) = \text{Zero}(C_1) \cup \text{Zero}(C_2),$$

with  $IS_1 = IS(C_1) \neq 0$ ,  $IS_2 = IS(C_2) \neq 0$ , where

$$C_1 = \{ \eta, u - \xi \},$$

$$C_2 = \{ \eta_{uu} - u + \xi, 1 + 2\xi_u, \eta_t + u\eta_x + \eta_{xx} + 2\eta\xi_x, 2\eta_{xu} - \xi_t + u\xi_x - 2\xi\xi_x - \xi_{xx}, \\ 2\eta_{tu} + 2\eta_x + u\xi_t + 4\eta_u\xi_x - u^2\xi_x + 2u\xi\xi_x + 2\xi_x^2 + \xi_{xt} + 2\xi\xi_{xx} + \xi_{xxx} \}.$$

**Step 3.6.** Since  $\text{Zero}(D/C) \neq \emptyset$ , one knows that the Burgers equation has a nontrivial non-classical symmetry by our algorithm and the corresponding reduced system of the DTEs  $D = 0$  is given by  $C_1 = 0$  and  $C_2 = 0$ .

From the above procedure, one has

$$\text{Zero}(D) = \text{Zero}(C) \cup \text{Zero}(C_1) \cup \text{Zero}(C_2). \tag{21}$$

Hence in order to determine explicit nontrivial non-classical symmetries of the Burgers equation, one needs to exactly determine the three components of the right hand side of (21). These three components are easier to compute than it is to solve the original DTEs  $D = 0$ .

The component  $\text{Zero}(C)$  corresponding to classical symmetries is easily determined from  $C' = 0$  (it is a linear system!) as follows:

$$\text{Zero}(C) = \left\{ \xi = \frac{c_1tx + c_2x + c_4t + c_5}{c_1t^2 + 2c_2t + c_3}, \eta = \frac{c_1(x - tu) - c_2u + c_4}{c_1t^2 + 2c_2t + c_3} \right\}.$$

For the non-classical case, solving the well-ordered system  $C_1 = C_2 = 0$ , one obtains the infinitesimal functions  $\xi$  and  $\eta$  as follows:

$$\text{Zero}(C_1) = \{ \xi = u, \eta = 0 \},$$

$$\text{Zero}(C_2) = \left\{ \xi = -\frac{1}{2}u + \alpha(x, t), \eta = \frac{1}{4}u^3 - \frac{1}{2}\alpha(x, t)u^2 - \beta(x, t)u + \gamma(x, t) \right\},$$

where  $\alpha(x, t)$ ,  $\beta(x, t)$ ,  $\gamma(x, t)$  satisfy the PDE system

$$\begin{cases} \alpha_t + \alpha_{xx} + 2\beta_x + 2\alpha\alpha_x = 0, \\ \beta_t + \beta_{xx} - \gamma_x + 2\beta\alpha_x = 0, \\ \gamma_t + \gamma_{xx} + 2\gamma\alpha_x = 0. \end{cases} \tag{22}$$

The three components in (21) correspond to the cases given in [15,20,29] obtained in different ways.

**Case**  $\tau = 0$ ,  $\xi \neq 0$ .

In this case, we let  $(\tau, \xi) = (0, 1)$  and have the invariant surface condition

$$u_x - \eta(x, t, u) = 0. \tag{23}$$

The conditional criterion implies only the single second-order equation  $D = 0$  in which the corresponding dps is

$$D = \{2\eta\eta_{xu} + \eta^2(\eta_{uu} + 1) + \eta_{xx} + \eta_t + u\eta_x\}$$

and the dps corresponding to classical symmetries in terms of  $\eta$  and  $\xi'$  is as follows

$$C'' = \{\eta_u, \eta_x, \eta_t + \eta^2, \xi'_u, \xi'_x, \eta\xi' - \xi'_t\}.$$

Hence

$$C = \{\eta_u, \eta_x, \eta_t + \eta^2\}.$$

It is easy to check that  $\text{Zero}(C) \subset \text{Zero}(D)$ , i.e.,

$$\text{Zero}(D/C) \neq \emptyset.$$

This shows the existence of nontrivial non-classical symmetries for the Burgers equation in the singular case  $(\tau, \xi) = (0, 1)$ . But we cannot get explicitly the solution  $\text{Zero}(D)$  for the nontrivial non-classical symmetry  $\mathcal{X}$ . A simple example of a solution is  $\eta = -u^2/2 \in \text{Zero}(D)$ .

**Remark 3.** The operator  $\mathcal{X}$  with  $(\tau, \xi) = (0, 1)$  is called a singular operator in the reduced form. Each solution  $\eta(t, x, u)$  of the determining equation  $D = 0$  corresponds to a family of solutions of the Burgers equation which are invariant with respect to this operator (see Corollary 5 and Theorem 3 in [32]). There exist an infinite number of solutions  $\eta(t, x, u)$  for the non-classical symmetries and a finite number of solutions for classical symmetries. Hence the existence of nontrivial non-classical symmetries is known from the general theorem in [32]. Our algorithm proves this from a different point of view.

**Example 2** (A nonlinear heat equation). As a second example, we consider the nonlinear heat equation

$$u_t = u_{xx} - u^3, \tag{24}$$

which arises in several physical and chemical applications [46].

The infinitesimal generators for classical symmetries of the PDE (24) are taken to be of the form (19).

**Case**  $\tau \neq 0$ .

In this case the infinitesimal generators for non-classical symmetries are the same as those for the previous example. Here one can show that the corresponding DTEs  $D' = D = 0$  are given by

$$D' = \{\eta'_x, \eta'_t, \eta' - u\eta'_u, \tau'_u, \tau'_x, u\tau'_t + 2\eta', \xi'_u, u\xi'_x + \eta', \xi'_t\},$$

$$D = \{\eta_t - \eta_{xx} + 2u^3\xi_x - \eta_u u^3 + \eta(3u^2 + 2\xi_x), \xi_{uu}, \xi_{xx} - 2\eta_{xu} - \xi_t + 3u^3\xi_u + 2\xi_u\eta - 2\xi_x\xi, 2\xi_{xu} - \eta_{uu} - 2\xi_u\xi\}.$$

The dchar-set of  $D'$  under the ranking  $\tau' < \xi' < \eta'$  is given by

$$C' = \{\eta'_x, \eta'_t, \eta' - u\eta'_u, \xi'_u, u\xi'_x + \eta', \xi'_t, \tau'_u, \tau'_x, u\tau'_t + 2\eta'\}.$$

Substituting the transformations  $\xi' = \tau'\xi$  and  $\eta' = \tau'\eta$  into  $C'$ , one gets

$$C' = \{\eta_x\tau' + \tau'_x\eta, \eta_t\tau' + \tau'_t\eta, \eta\tau' - u(\eta_u\tau' + \tau'_u\eta), \tau'_u, \tau'_x, u\tau'_t + 2\eta\tau', \xi_u\tau' + \tau'_u\xi, u(\xi_x\tau' + \tau'_x\xi) + \eta\tau', \xi_t\tau' + \tau'_t\xi\}.$$

Its dchar-set under the ranking  $\xi < \eta < \tau'$  is

$$C'' = \{\eta_x, \eta - u\eta_u, 2\eta^2 - u\eta_t, \tau'_u, \tau'_x, u\tau'_t + 2\eta\tau', \xi_u, u\xi_x + \eta, 2\eta\xi - u\xi_t\}.$$

Then

$$C = \{\eta_x, \eta - u\eta_u, 2\eta^2 - u\eta_t, \xi_u, u\xi_x + \eta, 2\eta\xi - u\xi_t\}.$$

Using our algorithm, one obtains

$$\text{Zero}(D/C) = \text{Zero}(C_1) \cup \text{Zero}(C_2),$$

where

$$C_1 = \{3u^3 + 2\eta, 9u^2 - 2\xi^2\}, \quad C_2 = \{u\xi^2 + 3\eta, \xi_u, \xi^2 - 3\xi_x, \xi_t\}.$$

Thus the nonlinear heat equation (24) has nontrivial non-classical symmetries which are found from solving the reduced system  $C_1 = C_2 = 0$ . In particular, one has the complete decomposition

$$\text{Zero}(D) = \text{Zero}(C_1) \cup \text{Zero}(C_2) \cup \text{Zero}(C).$$

One then finds the explicit nontrivial non-classical symmetries

$$\text{Zero}(C_1) = \left\{ \xi = \pm \frac{3}{\sqrt{2}}u, \eta = -\frac{3}{2}u^3 \right\}, \quad \text{Zero}(C_2) = \left\{ \xi = -\frac{3}{3c_1 + x}, \eta = -\frac{3u}{(3c_1 + x)^2} \right\},$$

where  $c_1$  is an arbitrary constant. This recovers the results in [1,19], but the latter is missed in [18,23].

**Case**  $(\tau, \xi) = (0, 1)$ .

Following the same procedure as in the previous case, we have DTEs  $D = 0$  and  $C'' = 0$  for NCSs and CSs respectively, where

$$\begin{aligned} D &= \{ \eta(3u^2 - 2\eta_{xu}) - \eta^2\eta_{uu} - \eta_{xx} + \eta_t - u^3\eta_u \}, \\ C'' &= \{ \eta_{uu}, \eta(3u^2 - 2\eta_{xu}) - \eta^2\eta_{uu} - \eta_{xx} + \eta_t - u^3\eta_u, \eta\eta_{tu} + \eta_u\eta_{xx} - \eta_t\eta_u + u^3\eta_u^2, \\ &\quad 2\eta\eta_{tx} + u^3\eta_{xx} - 2\eta\eta_{xxx} + 2\eta_x\eta_{xx} - u^3\eta_t - 2\eta_t\eta_x + u^6\eta_u - 3u^5\eta + 2u^3\eta_u\eta_x, \\ &\quad 3\eta_t\eta_{xx} - \eta\eta_{txx} + \eta\eta_{tt} - u^3\eta_u\eta_{xx} + 3u^2\eta\eta_{xx} - \eta_{xx}^2 + 2u^3\eta_t\eta_u - 3u^2\eta_t\eta - 2\eta_t^2 + 3u^5\eta_u\eta, \\ &\quad \xi'_u, \xi'_x, \eta\xi'_{tt} + 3u^2\xi'\eta_{xx} - 3u^2\eta_t\xi' + 3u^5\eta_u\xi' - 9u^4\eta\xi' \}. \end{aligned}$$

Hence

$$\begin{aligned} C &= \{ \eta_{uu}, \eta(3u^2 - 2\eta_{xu}) - \eta^2\eta_{uu} - \eta_{xx} + \eta_t - u^3\eta_u, \eta\eta_{tu} + \eta_u\eta_{xx} - \eta_t\eta_u + u^3\eta_u^2, \\ &\quad 2\eta\eta_{tx} + u^3\eta_{xx} - 2\eta\eta_{xxx} + 2\eta_x\eta_{xx} - u^3\eta_t - 2\eta_t\eta_x + u^6\eta_u - 3u^5\eta + 2u^3\eta_u\eta_x, \\ &\quad 3\eta_t\eta_{xx} - \eta\eta_{txx} + \eta\eta_{tt} - u^3\eta_u\eta_{xx} + 3u^2\eta\eta_{xx} - \eta_{xx}^2 + 2u^3\eta_t\eta_u - 3u^2\eta_t\eta - 2\eta_t^2 + 3u^5\eta_u\eta \}. \end{aligned}$$

It is easy to see that  $\text{Zero}(C) \subset \text{Zero}(D)$ . This shows the existence of nontrivial non-classical symmetries for the nonlinear heat equation (24) for the singular case  $(\tau, \xi) = (0, 1)$ . But we cannot obtain explicitly the solution set  $\text{Zero}(D)$  for a non-trivial non-classical  $\mathcal{X}$ . A simple example of a solution is given by  $\eta = cu^3 \in \text{Zero}(D)$  in terms of an arbitrary constant  $c$ .

**Example 3** (Coupled system of KdV-type equations). As a third example, we consider the coupled system of KdV-type equations given by

$$u_t + 3vv_x = 0, \quad v_t + 2v_{xxx} + 2uv_x + u_xv = 0, \tag{25}$$

which are found in the study of Kac–Moody algebras and in soliton theory [3]. Let the infinitesimal generator for a classical symmetry be  $\mathcal{X}' = \tau'\partial_t + \xi'\partial_x + \eta'\partial_u + \phi'\partial_v$ .

**Case**  $\tau \neq 0$ .

In this case, the DTEs for a non-classical infinitesimal generator  $\mathcal{X} = \partial_t + \xi\partial_x + \eta\partial_u + \phi\partial_v$  of (25) are given by  $D = 0$ , where

$$D = \left\{ \begin{aligned} &\phi_u, \phi_{vv}, \xi_v, \xi_u, \phi_{xv} - \xi_{xx}, \eta_t\xi - \xi_t\eta + (3v\phi_x + \eta_v\phi)\xi, \\ &(2\phi_{xxx} + \phi_t + 2u\phi_x + v\eta_x + 3\xi_x\phi)\xi + (v\eta_u + 2v\xi_x - v\phi_v + \phi)\eta, \\ &3(v\phi_v + \phi - v\eta_u - v\xi_x)\xi - \eta_v\xi^2 - 3v\xi_t, \\ &(6\phi_{xxv} - 2\xi_{xxx} - \xi_t + 4u\xi_x + v\eta_v - 3\xi_x\xi + 2\eta)\xi + 3(\eta_u + 2\xi_x - \phi_v)v^2 + 3v\phi \end{aligned} \right\}.$$

Using our algorithm, one has the decomposition

$$\text{Zero}(D/C) = \text{Zero}(C_1),$$

where

$$C = \{ \eta_v, \eta_x, \xi_v, \xi_u, \eta - u\eta_u, 3\eta^2 - 2u\eta_t, u\phi - v\eta, 2u\xi_x + \eta, 3\eta\xi - 2u\xi_t \},$$

is obtained from the DTEs for CSs and

$$C_1 = \left\{ \begin{aligned} &\eta_v, \phi_u, \xi_v, \xi_u, \phi_x, \phi - v\phi_v, v\eta_u(\xi - u) + v\eta - \xi\phi, \\ &2v\eta_t(u - \xi) + \eta\phi(3\xi - 4u) + v\eta^2, 2v\xi_x(u - \xi) + v\eta - \xi\phi, \\ &2u^2v\phi_t + 4u\xi(\phi^2 - v\phi_t) + \xi^2(2v\phi_t - 3\phi^2) + 2v\eta\phi(\xi - 2u) + v^2\eta^2, \\ &v^2(\eta^2 - 2\eta_x(u - \xi)^2) - v\eta\phi(u + \xi) + u\xi\phi^2, \\ &2v\xi_t(\xi - u) + \xi\phi(4u - 3\xi) - v\eta\xi \end{aligned} \right\}.$$

Hence the coupled KdV-type system (25) has nontrivial non-classical symmetries which are determined from Zero(C<sub>1</sub>). They are given by the three sets of infinitesimals

$$\begin{aligned} \xi &= \frac{1}{3}x(\tanh(c_1t - c_2) + 1)c_1 + e^{c_1t} \operatorname{sech}(c_1t - c_2)c_3, \\ \eta &= \left( \frac{1}{3}x(\tanh(c_1t - c_2) + 1)c_1 + e^{c_1t} \operatorname{sech}(c_1t - c_2)c_3 - \frac{2}{3}u(\tanh(c_1t - c_2) + 1) \right) c_1, \\ \phi &= \frac{1}{3}v(1 - 2 \tanh(c_1t - c_2))c_1; \end{aligned} \tag{26}$$

$$\begin{aligned} \xi &= \frac{1}{3}x(\coth(c_1t - c_2) + 1)c_1 + e^{c_1t} \operatorname{csch}(c_1t - c_2)c_3, \\ \eta &= \left( \frac{1}{3}x(\coth(c_1t - c_2) + 1)c_1 + e^{c_1t} \operatorname{csch}(c_1t - c_2)c_3 - \frac{2}{3}u(\coth(c_1t - c_2) + 1) \right) c_1, \\ \phi &= \frac{1}{3}v(1 - 2 \coth(c_1t - c_2))c_1; \end{aligned} \tag{27}$$

$$\begin{aligned} \xi &= c_2e^{2c_1t}, \\ \eta &= c_1c_2e^{2c_1t}, \\ \phi &= c_1v, \end{aligned} \tag{28}$$

where  $c_1 \neq 0$ . In [3], only the restricted solution (26) with  $c_3 = 0$  and the solution (28) were found. Here the complete set of nontrivial non-classical symmetries for (25) is obtained.

**Case**  $(\tau, \xi) = (0, 1)$ .

In this case, we have the DTEs  $D = 0$  for non-classical symmetries, where

$$\begin{aligned} D &= \{ \eta_t - \eta(\eta_v(4\phi\phi_{uv} + 4\phi_{xu} + 2\phi_u\phi_v + 2\eta_u\phi_u + v) - 3v\phi_u) - 2\eta_v\eta^2\phi_{uu} - 4\eta_v\phi\phi_{xv} - 2\eta_v\phi_{xx} \\ &\quad 3v\phi_x - 2\eta_v\phi^2\phi_{vv} - 2\phi_u\eta_v\eta_x - 3v\eta_u\phi - 2u\eta_v\phi - 2\phi_u\eta_v^2\phi - 2\eta_v\phi_v\phi_x - 2\eta_v\phi_v^2\phi + 3v\phi_v\phi + 3\phi^2; \\ &\quad \eta(\eta_v(-2\eta_{tu} + 6v\phi\eta_{uu} + 6v\eta_u\phi_u + 6v\phi_u\phi_v - 18\phi_u\phi + 3v^2) + 6v\phi_u\eta_{xv} + 6v\phi_x\eta_{uv} - 6v\eta_u\phi\eta_{uv} \\ &\quad + 6v\phi_u\phi\eta_{vv} + 6v\phi_v\phi\eta_{uv} + 6\phi^2\eta_{uv} - 9v^2\phi_u) + \eta_t(2\eta\eta_{uv} + 2\eta_{xv} + 2\phi\eta_{vv} + 2\eta_v\phi_v - 3v) - 2\eta_v\eta_{tx} \\ &\quad - 2\eta_v\phi\eta_{tv} + 6v\eta_v\phi\eta_{xu} - 6v\eta_u\phi\eta_{xv} + 6v\eta_v\phi^2\eta_{uv} - 6v\eta_u\phi^2\eta_{vv} + 6v\phi_u\eta^2\eta_{uv} + 6\phi^2\eta_{xv} + 6v\phi_v\phi\eta_{xv} \\ &\quad + 6v\phi_x\phi\eta_{vv} + 6v\phi_x\eta_{xv} + 6\phi^3\eta_{vv} + 6v\phi_v\phi^2\eta_{vv} - 2\phi_t\eta_v^2 + 9v^2\eta_u\phi + 6v\eta_u\eta_v\phi_x + 6\eta_u\eta_v\phi^2 + 6uv\eta_v\phi \\ &\quad + 6v\phi_u\eta_v^2\phi - 9v^2\phi_x - 9v^2\phi_v\phi - 18\eta_v\phi_x\phi + 6v\eta_v\phi_v\phi_x - 12\eta_v\phi_v\phi^2 + 6v\eta_v\phi_v^2\phi - 9v\phi^2 \} \end{aligned}$$

and

$$C = \{ \eta, \phi \}.$$

Obviously,  $\text{Zero}(D/C) \neq \emptyset$ . Hence in this case Eqs. (25) admit nontrivial non-classical symmetries.

**Example 4** (ALWW equations). As a final example, we apply our algorithm to the approximate long water wave (ALWW) equations [31]:

$$u_t - uu_x - v_x + \alpha u_{xx} = 0, \quad v_t - (uv)_x - \alpha v_{xx} = 0. \tag{29}$$

The infinitesimal generators for classical and non-classical symmetries are taken to be of the same form as for Example 3.

**Case**  $\tau \neq 0$ .

Here one gets the decomposition

$$\text{Zero}(D/C) = \emptyset \iff \text{Zero}(D) = \text{Zero}(C)$$

for the DTEs  $D = 0$  for a non-classical symmetry infinitesimal generator of the form  $\mathcal{X} = \partial_t + \xi\partial_x + \eta\partial_u + \phi\partial_v$  for the PDE system (29), where

$$C = \{ \phi_u, \phi_x, \phi_t, \phi - v\phi_v, \eta_v, \eta_x, \eta_t, \phi - 2v\eta_u, \xi_v, \xi_u, 2v\xi_x + \phi, u\phi - 2v\xi_t - 2v\eta \},$$

is the dchar-set for classical symmetries of the system (29) with  $IS \neq 0$ . Hence the PDE system (29) has no nontrivial non-classical symmetries (Step 3.7 in our algorithm). In fact,  $\text{Zero}(C)$  only yields infinitesimal generators  $\mathcal{X} = \tau\partial_t + \xi\partial_x + \eta\partial_u + \phi\partial_v$  for classical symmetries of the PDE system (29) with

$$\tau = -2c_1 t + c_3, \quad \xi = -c_1 x - c_2 t + c_4, \quad \eta = c_1 u + c_2, \quad \phi = 2c_1 v,$$

for arbitrary constants  $c_1, c_2, c_3$  and  $c_4$ .

**Case**  $(\tau, \xi) = (0, 1)$ .

Similarly, here we have differential polynomial systems

$$\begin{aligned} D = & \{ \eta(2\alpha\phi\eta_{uv} + 2\alpha\eta_{xu} + \eta_v(2\alpha\phi_u + v) - \phi_u) + \eta^2(\alpha\eta_{uu} - 1) + 2\alpha\phi\eta_{xv} + \alpha\phi^2\eta_{vv} + \alpha\eta_{xx} + \eta_t - u\eta_x \\ & + \eta_u\phi + 2\alpha\eta_v\phi_x + 2\alpha\eta_v\phi_v\phi - \phi_v\phi - \phi_x; \eta(2\phi(\alpha\phi_{uv} + 1) + 2\alpha\phi_{xu} + \eta_u(2\alpha\phi_u + v) - v\phi_v) + \alpha\eta^2\phi_{uu} \\ & + 2\alpha\phi\phi_{xv} + \alpha\phi^2\phi_{vv} + \alpha\phi_{xx} - \phi_t + \eta_x(2\alpha\phi_u + v) + 2\alpha\phi_u\eta_v\phi + u\phi_x - \phi_u\phi + v\eta_v\phi \}, \\ C'' = & \{ \eta_v, \eta_u, \eta_x, \eta^2 - \eta_t, \phi, \xi_v, \xi_u, \xi_x, \xi_t + \eta\xi \} \end{aligned}$$

for classical and non-classical symmetries. Consequently,

$$C = \{ \eta_v, \eta_u, \eta_x, \eta^2 - \eta_t, \phi \}.$$

Thus, obviously  $\text{Zero}(C/C) \neq \emptyset$ . Hence in this case Eqs. (29) admit nontrivial non-classical symmetries.

## 5. New ansatzes for invariant solutions

In this section as an extended application of our algorithm, we obtain new ansatzes for the invariant solutions of the coupled-KdV-type equations (25) through reduction of the equations into ordinary differential equations using the newly obtained nontrivial non-classical symmetries (26) and (27).

Using the nontrivial NNCS (26) and solving the corresponding invariant surface equations

$$\xi u_x + u_t - \eta = 0,$$

$$\xi v_x + v_t - \phi = 0,$$

one obtains invariant solutions of (25) expressed in terms of the invariants  $U, V$  and  $\zeta$  by

$$u(t, x) = \left( \frac{1}{3} c_1 \zeta e^{2c_1 t - c_2} + 3U(\zeta) \right) \sqrt[3]{(2e^{-c_1 t} \text{sech}(c_1 t - c_2))^2}, \quad (30)$$

$$v(t, x) = V(\zeta) \sqrt[3]{e^{c_1 t} \text{sech}^2(c_1 t - c_2)}, \quad (31)$$

$$\zeta = (x + 3e^{c_2} c_3 / c_1) \sqrt[3]{2e^{-c_1 t} \text{sech}(c_1 t - c_2)}.$$

Inserting these expressions into (25), one obtains the reduced coupled system of ODEs in  $U(\zeta)$  and  $V(\zeta)$  given by

$$\begin{aligned} VV' + k_1 \zeta U' + 2k_1 U + k_2 \zeta &= 0, \\ 2V^{(3)} + 3VU' + 6UV' + k_3 V &= 0, \end{aligned} \quad (32)$$

where  $k_1 = -\frac{c_1 e^{-c_2}}{3\sqrt[3]{2}}$ ,  $k_2 = \frac{c_1^2}{9\sqrt[3]{2}}$  and  $k_3 = c_1 e^{c_2}$ . Note that in (32),  $k_2 > 0$  and  $k_1, k_2$ , and  $k_3$  satisfy the relation  $k_1 * k_3 = -3k_2$ . It appears that this system can only be solved numerically.

The ansatzes (30) and (31) for  $u(t, x)$  and  $v(t, x)$  cannot be derived through classical symmetry reduction. Hence we obtain a new class of solutions of (25).

Using the NNCS (27) and solving the corresponding invariant surface condition equations, one obtains the corresponding invariant solutions of (25) expressed in terms of the invariants  $U, V$  and  $\zeta$  by

$$u(t, x) = \left( \frac{1}{6} \zeta e^{2c_1 t - c_2} + U(\zeta) \right) \sqrt[3]{(e^{-c_1 t} \text{csch}(c_1 t - c_2))^2},$$

$$v(t, x) = V(\zeta) \sqrt[3]{e^{c_1 t} \text{csch}^2(c_1 t - c_2)},$$

$$\zeta = (x + 3e^{c_2} c_3 / c_1) c_1 \sqrt[3]{e^{-c_1 t} \text{csch}(c_1 t - c_2)}.$$

Inserting these expressions into (25), one obtains the coupled system of ODEs in  $U(\zeta)$  and  $V(\zeta)$  given by

$$\begin{aligned} 18VV' - 2\zeta U' - 4U - \zeta &= 0, \\ 4c_1^2 V^{(3)} - e^{c_2} V + 2VU' + 4UV' &= 0. \end{aligned}$$

This ansatz for  $u(t, x)$  and  $v(t, x)$  cannot be derived through classical symmetry reduction. Thus, we obtain another new class of solutions of (25).



## 6. Conclusions and discussion

Using the so-called differential characteristic set decomposition of the determining system for classical and non-classical symmetries, we have introduced an algorithmic method to decide on the existence of a nontrivial non-classical symmetry of a PDE without solving the determining equations for an NNCS. The decomposition is realized by using the differential characteristic set method. Furthermore, we also have presented an algorithm for reducing the determining equations into simpler forms (dchar-sets). Our method partially answers the open problem posed by Clarkson and Mansfield [21] concerning the existence of non-classical symmetries. Our algorithms provide an alternative way to study the computation of non-classical symmetries for a given PDE. As illustrative examples, we determined non-classical symmetries of some evolution equations including the Burgers equation, a nonlinear heat equation, a system of coupled KdV-type equations, and the approximate long water wave equations. In particular, we found new non-classical symmetries for the considered coupled KdV-type system, and correspondingly exhibited new exact solutions to the equations.

As far as the authors know, this is the first paper to decide systematically on the existence of a nontrivial non-classical symmetry of a PDE without solving the corresponding non-classical symmetry determining equations.

The dchar-set method was established by the Chinese mathematician Wentsun Wu in the 1970s, based on Ritt's theory [42] originally for mechanical (automatic) theorem-proving purposes. It also has become a fundamental algorithmic theory in algebraic geometry together with the Gröbner base algorithm. The method has been applied to many fields including mechanical theorem proving, optimization problems, surface-fitting problems in CAGD, Bar Linkage Design, etc. [50]. The differential analogue of Wu's method was proposed in the 1980s [49].

The concept of dchar-sets is the key ingredient in both Ritt's and Wu's works. In Ritt's work, the concept of dchar-set is defined for an algebraic ideal generated by a dps, whereas here, as in Wu's method, the dchar-set is simply defined for the dps (basis of an ideal). In particular, the main focus of Wu's method is to directly deal with the zero set and reduction of a dps  $DP$  [50]. The fundamental decomposition (7) or (8) provided by the method is the most critical tool of our algorithm. The analysis of zero sets of a dps in the method gives rise to the fundamental notion of a dchar-set and further principles under the names: well-ordering principles, zero-decomposition theorems, variety-decomposition theorems, etc. Developing the theory, Wu put forward a well-known mechanical proving algorithm for a geometric theorem [50]. We found that Lie's criterion on the invariance of a PDE can be transformed to the mechanical proving algorithm which gives an alternative way to deal with the symmetry computation, symmetry classification etc. [11,12,14,15].

There are several alternative algorithms for dealing with differential polynomial systems that have been developed. Corresponding Maple programs have been implemented such as Reid's rif-algorithm and package *rifsimp* [40,41], Mansfield's differential Gröbner basis algorithm and package *diffgrob2* [33], Boulier's Rosenfeld-Gröbner algorithm and package *difffalg* [9], and the Thomas method and package [27]. These algorithms and their packages are now the commonly used computational tools for differential algebra fields. Mainly, these methods can be used to perform the elimination and reduction computations for dealing with a given dps.

A consistent theme in these algorithms is the idea that one first reduces the dps under consideration to some kind of 'standard form' (e.g. *Gröbner basis*, *dchar-set*, *rif-form*, *involution form* [45], *Simple*, etc.) and then integrates or uses the obtained 'well-ordering' system. However, the procedures for obtaining the reduced 'standard form' are different. An obtained decomposition (if it exists), such as (7) or (8), may admit different properties. We believe that these methods, which are analogues of the decompositions (7) and (8) with irreducible properties, can be used for solving the symmetry-decision problems. We will consider these in a future study. In this paper, we have focused on giving an alternative example method for solving symmetry-decision problems.

In practice, the major problem with the dchar-set algorithm is that the expressions swell in their computations which may lead to lack of success. This is also a common problem of symbolic computation. However, for a system where the computations can be completed within reasonable limits, i.e., the length of obtained expressions is small enough to be meaningful, the output is extremely useful. For such a system, a comparison of the determining equations for nontrivial non-classical symmetries and its dchar-set demonstrates this point; cf. examples given in Section 4.

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